Stochastic Control in Market Making

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Oxford Mathematics

Stochastic Control Problem

- state process $X = (X_t)_{t \in T}$;
- have control α_t exert to the trajectory of X
- then we get the controlled diffusion process X

$$\mathrm{d}X_t = b(X_t, \alpha_t)\mathrm{d}t + \sigma(X_t, \alpha_t)\mathrm{d}W_t, X_0 = x \in \mathcal{R}^n$$

, where W_t is $d-$dimensional BM, control process α_t lying on control set $A \subset \mathcal{R}^m$. $b : \mathcal{R}^n \times A \mapsto \mathcal{R}^n, \sigma : \mathcal{R}^n \times A \mapsto \mathcal{R}^{n \times d}$

- Finite Horizon Objective:

$$J(x; \alpha) := \mathbf{E}[\int_0^T f(t, X_t, \alpha_t)\mathrm{d}t + F(X_T)]$$

- Value Function:

$$u := \sup_{\alpha \in A} J(x; \alpha)$$

Market Making Basics

- Definitions
  - Market Orders (MO)
  - Limit Orders (LO): Passive Quoting, resting on book
  - Limit Order Book (LOB): Has bid and ask side, consisting of LOs
  - Mid Price: Avg of Bid and Ask Price
  - Spread: the difference between ask and bid
- Market Maker (MM): Market Maker quoting on both bid and ask side, providing liquidity to the market and charging the spread
  - MM wants to minize risk while maximize the profit.

Into the Market I

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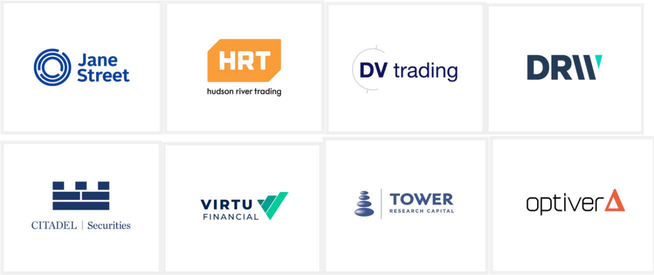

Figure: Well known MMs

Into the Market II

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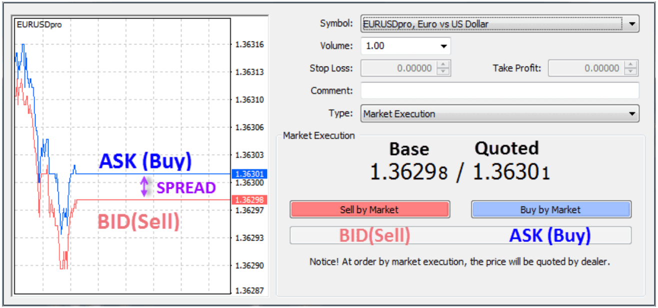

Figure: A trading infra with Bid and Ask

Into the Market III

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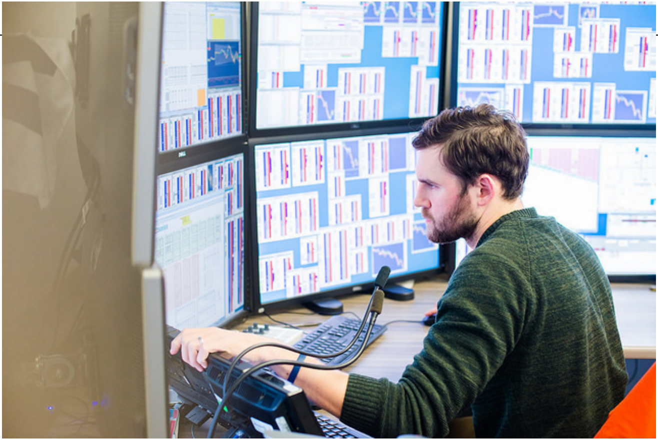

Figure: DV Trading

Market Making Problem I

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- Mid Price Process S_t

$$S_t = S_0 + \sigma W_t$$

- MM posting depth: $\delta^{\pm}$, LO on bid side: $S - \delta^{-}$, on ask side: $S + \delta^{+}$
- MO Arrival Counting Process: $M^{\pm} \sim Pois(\lambda^{\pm})$
- LO filling Counting Process: $N^{\pm} \sim Exp(\kappa^{\pm})$:

$$\mathrm{d}N^{\pm} = \exp(-\kappa^{\pm}\delta_t^{\pm})\mathrm{d}M^{\pm}$$

- MM Cash Process:

$$\mathrm{d}X_t^{\delta} = (S_{t-} + \delta_t^{+})\mathrm{d}N_t^{\delta,+} - (S_{t-} - \delta_t^{-})\mathrm{d}N_t^{\delta,-}$$

Market Making Problem II

- Inventory Process: $Q_t^\delta = N_t^{\delta,-} - N_t^{\delta,+}$
- MM's performance:

$$H^\delta(t,x,S,q) = \mathbf{E}_{t,x,S,q}[X_T + Q_T(S_T - \alpha Q_T) - \phi \int_t^T (Q_u)^2 d_u]$$

- Objective:

$$H(t,x,S,q) = \sup_{\delta\pm} H^\delta(t,x,S,q)$$

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- The MM use random control δ during $[t, t+\epsilon)$
- The MM use optimal control $\hat{\delta}$ during $[t+\epsilon, T]$

Then we obtain the following inequality

$$H(t,x,S,q) \geq E[-\int_t^{t+\epsilon} Q_u^2 \mathrm{d}u + H(t+\epsilon, x_{t+\epsilon}, S_{t+\epsilon}, q_{t+\epsilon})] \quad (1)$$

Hamilton–Jacobi–Bellman Equation I

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$$
\begin{aligned}
H(t+\epsilon, x_{t+\epsilon}, S_{t+\epsilon}, q_{t+\epsilon}) &= H(t,x,S,q) \\
&+ \int_t^{t+\epsilon} \partial_t H + \frac{\sigma^2}{2}\partial_{SS} H \mathrm{d}u + \int_t^{t+\epsilon} \sigma \partial_S H \mathrm{d}W_u \\
&+ \int_t^{t+\epsilon} (H(u, x + (S+\delta^+), S, q-1) - H) \mathrm{d}N^{\delta,+} \\
&+ \int_t^{t+\epsilon} (H(u, x - (S-\delta^-), S, q+1) - H) \mathrm{d}N^{\delta,-}
\end{aligned}
\tag{2}
$$

Hamilton–Jacobi–Bellman Equation II

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Inject back to Eqn(1), take the sup we get:

$$\begin{aligned} 0 =& \partial_t H + \frac{1}{2}\sigma^2 \partial_{SS} H - \phi q^2 \\ &+ \lambda^+ \sup_{\delta^+}[e^{-\kappa^+\delta^+}(H(t, x + (S + \delta^+), q - 1, S) - H)] \\ &+ \lambda^+ \sup_{\delta^-}[e^{-\kappa^-\delta^-}(H(t, x - (S - \delta^-), q + 1, S) - H)] \end{aligned} \tag{3}$$

with the terminal condition:

$$H(T, x, S, q) = x + q(S - \alpha q)$$

Optimal Controls

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For the following ansatz for H

$$H(t, x, q, S) = x + qS + h(t, q)$$

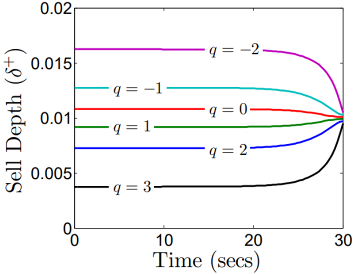
We have optimal controls:

$$\delta^+ = \frac{1}{\kappa^+} - h(t, q-1) + h(t, q)$$

and

$$\delta^- = \frac{1}{\kappa^-} - h(t, q+1) + h(t, q)$$

Property of the Solution, $\phi = 0.01$

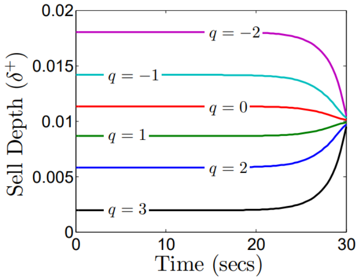


(a) $\phi = 0.001$

Figure: The optimal depths as a function of time for various inventory levels and $T = 30$. The remaining model parameters are: $\lambda^{\pm} = 1$, $\kappa^{\pm} = 100$, $\overline{q} = -\underline{q} = 3$, $\phi = 0.001$ and $\phi = 0.02$, $\alpha = 0.0001$, $\sigma = 0.01$, $S_0 = 100$.

Figure: Solution, credit to Prof. Jan Obloj

Property of the Solution, $\phi = 0.02$



(a) $\phi = 0.02$

Figure: The optimal depths as a function of time for various inventory levels and $T = 30$. The remaining model parameters are: $\lambda^{\pm} = 1$, $\kappa^{\pm} = 100$, $\overline{q} = -\underline{q} = 3$, $\phi = 0.001$ and $\phi = 0.02$, $\alpha = 0.0001$, $\sigma = 0.01$, $S_0 = 100$.

Figure: Solution, credit to Prof. Jan Obloj

Drawbacks & Frontiers

- Mid Price is not a good measure
  - Fair Price which measures instruments correlation
- Fair Price is hard to formulate
  - Incorporate Alpha signals into Price Formulation
- No Aggressive Fill possibility
- Hawkes Process instead of Poisson Process
- Poor Property leads to numerical solution, while solving high dimensional PDE is time-consuming

Jane Street sue LME

US trading firm Jane Street sues LME for nickel trade chaos

Lawsuit comes a day after similar action by Elliott against metals exchange over March trade halt

The London Metal Exchange's owner has said the claims are 'without merit' © Jason Alden/Bloomberg

William Langley in Hong Kong JUNE 7 2022 30

Jane Street, a top Wall Street market maker, has followed US hedge fund Elliott Management and launched a lawsuit against the London Metal Exchange over the nickel short squeeze fiasco.

Figure: Jane Street Sue London Metal Exchange